UNIT – III
FOURIER SERIES

Fourier series

Suppose that a given function $f(x)$ defined in $[-\pi,\pi]$ (or) $[0,2\pi]$ (or) in any other interval can be expressed as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

The above series is known as the Fourier series for $f(x)$ and the constants $a_0,a_n,b_n(n=1,2,3-----)$ are called Fourier coefficients of $f(x)$

Periodic Function:-

A function $f(x)$ is said to be periodic with period $T>0$ if for all $x, f(x+T)=f(x)$,and T is the least of such values

Example:- (1) $\sin x=\sin(x+2\pi)=\sin(x+4\pi)=-----$ the function $\sin x$ is periodic with period 2π. There is no positive value T, $0<T<2\pi$ such that $\sin(x+T)=\sin x\forall\ x$

(2) $The\ period\ of\ tan\ x\ is\ \pi$

(3) The period of sin$nnx\ is\ \frac{2\pi}{n}\quad i.e\ sinnx=sinn\left(\frac{2\pi}{n}+x\right)$

Euler's Formulae:-

The Fourier series for the function $f(x)$ in the interval $C\le x\le C+2\pi$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Where $a_0=\frac{1}{\pi}\int_C^{C+2\pi}f(x)dx$

$a_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\cos nx.dx$ and

$b_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\sin nx.dx$

These values of a_0,a_n,b_n are known as Euler's formulae

Corollary:- If $f(x)$ is to be expanded as a Fourier series in the interval $0\le x\le 2\pi$, put $C=0$ then the formulae (1) reduces to

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx \qquad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx.dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx.dx$$

Corollary 2:- If $f(x)$ is to expanded as a Fourier series in $[-\pi,\pi]$ put $c=-\pi$, the interval becomes $-\pi\le x\le\pi$ and the formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

Functions Having Points of Discontinuity :-

In Euler's formulae for a_0, a_n, b_n it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fourier series

Let $f(x)$ be defined by

$$f(x) = \phi(x), c < x < x_0$$
$$= \phi(x), x_0 < x < c + 2\pi$$

Where x_0 is the point of discontinuity in $(c, c+2\pi)$ in such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\,dx + \int_{x_0}^{c+2\pi} \phi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\cos nx.dx + \int_{x_0}^{c+2\pi} \phi(x)\cos nx.dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{c}^{x_0} \phi(x)\sin nx.dx + \int_{x_0}^{c+2\pi} \phi(x)\sin nx.dx\right]$$

Note :-

(i) $\int_{-\pi}^{\pi} \cos mx \cos nx.\,dx = \begin{cases} 0\ for\ m \neq n \\ \pi, for\ m = n > 0 \\ 2\pi, for\ m = n > 0 \end{cases}$

(ii) $\int_{-\pi}^{\pi} \sin mx \sin nx.\,dx = \begin{cases} 0\ for\ \ m = n = 0 \\ \pi, for\ m \neq n\ > 0 \end{cases}$

Problems:-

Fourier Series in $[-\pi, \pi]$

1. ***Express*** $f(x) = x$ ***as Fourier series in the interval*** $-\pi < x < \pi$

Sol : Let the function x be represented as a Fourier series

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \qquad (\because \text{x is odd function})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.\,dx$$

$= 0$ ($x\cos nx$ is odd function and $\cos nx$ is even function)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x)\sin nx.\,dx = = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\ \sin nx.\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{-\pi}^{\pi} x\ \sin nx.\,dx\right] \quad [\because x\sin nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (0+0)\right]$$

$(\because \sin n\pi = 0, \sin 0 = 0)$

$$= -\frac{2}{n}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3\ldots..$$

Substituting the values of a_0, a_n, b_n in (1), We get

$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots.\right]$$

2. **Express $f(x) = x - \pi$ as Fourier series in the interval $-\pi < x < \pi$**

Sol:

Let the function $x - \pi$ be represented by the Fourier series

$$f(x) = x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1) \qquad \text{Then}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\pi)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx = \pi\int_{-\pi}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[0 - \pi.2\int_{0}^{\pi} dx\right] \quad (\because x \text{ is odd function})$$

$$= \frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right] \qquad = -2(\pi - 0) = -2\pi$$

$$\text{and } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.\,dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.\,dx$$

$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.\,dx - \pi\int_{-\pi}^{\pi}\cos nx.\,dx\right] \qquad = \frac{1}{\pi}\left[0 - 2\pi\int_0^{\pi}\cos nx.dx\right]$$

($x\cos nx$ is odd function and $\cos nx$ is even function)

$$\therefore\ a_n = -2\int_0^{\pi} \cos nx.\,dx = -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{-2}{n}(\sin n\pi - \sin 0) = \frac{-2}{n}(0-0) = 0 \ \text{ for } n = 1,2,3 \ldots\ldots\ldots.$$

$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\ \sin nx.\,dx - \pi\int_{-\pi}^{\pi}\sin nx.\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{-\pi}^{\pi} x\ \sin nx.\,dx - \pi(0)\right] \quad [\because x\sin nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (0+0)\right] \ (\because \sin n\pi = 0)$$

$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3.....$$

Substituting the values of a_0, a_n, b_n in (1), We get

$$\therefore\ x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots.\right]$$

3. Find the Fourier series to represent the function e^{-ax} *from* $-\pi \le x \le \pi$.

Deduce from this that $\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} ------\right]$ Sol. Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} = \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$\therefore \frac{a_0}{2} = \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\frac{1}{a\pi} = \frac{\sinh a\pi}{a\pi}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx.\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{-ax}\cos bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\therefore\ a_n = \frac{1}{\pi}\left\{\frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0) - \frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0)\right\}$$

$$= \frac{a}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi})\cos n\pi = \frac{2a\cos n\pi\ \sinh a\pi}{\pi(a^2+n^2)}$$

$$= \frac{(-1)^n 2a\ \sinh a\pi}{\pi(a^2+n^2)} (\because \cos n\pi = (-1)^n)$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx.\,dx$

$$\left[\because \int e^{ax}\sin bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}(0 - n\cos n\pi) - \frac{e^{a\pi}}{a^2+n^2}(0 - n\cos n\pi)\right]$$

$$= \frac{n\cos n\pi(e^{a\pi}-e^{-a\pi})}{\pi(a^2+n^2)} = \frac{(-1)^n 2n\ \sinh a\pi}{\pi(a^2+n^2)}$$

Substituting the values of $\frac{a_0}{2}, a_n$ and b_n in (1) we get

$$e^{-a\pi} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2a\ \sinh a\pi}{\pi(a^2+n^2)}\cos nx + (-1)^n 2n\frac{\sinh a\pi}{\pi(a^2+n^2)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{a}\left\{\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + ..\right)\left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2}\ ...\right)\right\} \text{——(2)}$$

Deduction: –

Putting $x=0$ and $a=1$ in (2), we get

$$1 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - -\right] \Rightarrow \frac{\pi}{\sinh\pi} = 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - -\right)$$

4. Find the Fourier Series of $f(x) = x + x^2, -\pi < x < \pi$ and hence deduce the series

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + - - - - = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + - - = \frac{\pi^2}{12}$

Sol: Let $x + x2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

To find ao $= \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x + x^2)\,dx = \frac{1}{\Pi}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)_{\pi}^{-\pi} = \frac{2}{3}\pi^2$

To find an $= \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x + x^2)\cos nx\,dx$

$$= \frac{1}{\pi}\left[(x + x^2)\frac{\sin nx}{n} - (1 + 2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\pi}\left[(x + x^2)\frac{\sin nx}{n} - (1 + 2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\Pi n^2}[(1 + 2x)(\cos nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\Pi n^2}[(1 + 2\pi)(\cos n\pi) - (1 - 2\pi)(\cos n\pi)]$$

$$= \frac{1}{\Pi n^2}(4\pi\ \cos n\pi) = \frac{4}{n^2}(-1)^n$$

To find bn $= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{-\cos nx}{n} - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\frac{-\cos n\Pi}{n} - 0 + 2\left(\frac{\cos n\Pi}{n^3}\right)\right] - \left[(\pi+\pi^2)\frac{-\cos n\pi}{n} - 0 + 2\left(\frac{\cos n\pi}{n^3}\right)\right] = -\frac{2}{n}(-1)^n$$

Substituting in (1), the required Fourier series is,

$$x + x^2 = \frac{\pi^2}{3} - 4(\text{Cos}\,x - \text{Cos}\,\frac{2x}{4} + \text{Cos}\,\frac{3x}{9} + ..) + 2(\text{Sin}\,x - \text{Sin}\,\frac{2x}{4} + \text{Sin}\,\frac{3x}{9} + \cdots)$$

5. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}$ Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ------ = \frac{\pi^2}{8}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$ then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_0^{\pi} xdx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{-\pi^2}{2}\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx.dx + \int_0^{\pi} x\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{\pi n^2}\right]$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}[(-1)^n_{-1}]$$

$$a_1 = \frac{-2}{1^2.\pi}, a_2 = 0, a_3 = \frac{-2}{3^2.\pi}, a_4 = 0, a_5 = \frac{-2}{5^2.\pi} ---$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx + \int_0^{\pi} x\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$b_1 = 3, b_2 = \frac{-1}{2}, b_3 = 1, b_4 = \frac{-1}{4} \text{ and so on}$$

Substituting the values of a_0, a_n *and* b_n in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ---\right) + \left(3\sin x - \frac{\sin 3x}{2} + \frac{3\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots\right) \ldots (2)$$

<u>Deduction:-</u>

Putting $x=0$ in (2), we obtain $f(0) = \frac{-\pi}{4} - \frac{2}{4}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right)$ —— .... (3)

Now $f(x)$ is discontinuous at $x=0$

$f(0-0) = -\pi$ *and* $f(0+0) = 0$

$f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = \frac{-\pi}{2}$

Now (3) becomes $\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ---\right) \rightarrow \frac{1}{1^2} + \frac{1}{3^2} + {}^{1}/_{5^2} + \cdots = \frac{\pi^2}{8}$

6. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{bmatrix}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1)$ then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)dx + \int_0^{\pi}\pi dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \pi(x)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2+\pi^2\right] = \frac{1}{\pi}[0] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx + \int_0^{\pi}\pi\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \pi\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}(0) \quad (\because \sin 0 = 0, \sin n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx + \int_0^{\pi}\pi\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\pi\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}(\cos n\pi - cos 0)\right]$$

$$= \frac{1}{n}(2-2\cos n\pi) = \frac{1}{n}(2-2(-1)^n) = \begin{cases} 0 \text{ when } n \text{ is even} \\ \frac{4}{n} \text{ when } n \text{ is odd} \end{cases}$$

Substituting the values of a_0, a_n and b_n in (1), we get $f(x) = \sum_{n=1}^{\infty}\frac{4}{n}\sin(nx)$ *where n is odd*

$$f(x) = 4\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) \pm --\right)$$

Fourier Series for $f(x)$ in $[0, 2\pi]$

1. ***Obtain the Fourier series for the function*** $f(x) = e^x$ ***from*** $x = [0, 2\pi]$

Sol: Let $e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1)$

Then $ao = \frac{1}{\pi}\int_0^{2\Pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\Pi} e^x\,dx = \frac{1}{\pi}(e^x)_0^{2\pi} = \frac{1}{\pi}(e^{2\Pi} - 1)$

and $an = \frac{1}{\pi}\int_0^{2\Pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\Pi} e^x \cos nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi} = \frac{e^{2\pi}-1}{\pi(1+n^2)}$$

$$Finally \quad bn = \frac{1}{\pi}\int_0^{2\Pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\Pi} e^x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}$$

$$Hence\ e^x = \frac{e^{2\pi}-1}{\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx$$

$$= \frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}\right]$$

This is the required Fourier series.

2. *Obtain the Fourier series to represent the function*

$$\boldsymbol{f(x) = kx(\pi - x)\ in\ 0 < x < 2\pi\ Where\ k\ is\ a\ constant.}$$

Sol: − Given $f(x) = kx(\pi - x)$ *in* $0 < x < 2\pi$ *fourier series of the function* $f(x)$

$$f(x) = kx(\pi - x)\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx ----------(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\,dx = \frac{k}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi} = -\frac{2\pi^2 k}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\cos nx\, dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{0 + \frac{-3\pi}{n^2}\cos 2n\pi + 0\right\} - \left\{0 + \frac{\pi}{n^2} + 0\right\}\right] = \frac{k}{\pi}\left(\frac{-4\pi}{n^2}\right) = -\frac{4k}{n^2}\ (n \neq 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\sin nx\, dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{\frac{2\pi^2}{n} + 0 - \frac{2}{n^3}\right\} - \left\{0 + 0 - \frac{2}{n^3}\right\}\right] = \frac{2k\pi}{n}$$

put the values of a_0, a_n, b_n *in (1) we get*

$$f(x) = -\frac{\pi^2 k}{3} - 4k\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 2k\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

3.. Find the fourier series expansion of the function $\boldsymbol{f(x) = \frac{(\pi - x)^2}{4}}$ **in the interval** $\boldsymbol{0 < x < 2\pi}$

Sol:

$$\text{Given } f(x) = \frac{(\pi - x)^2}{4}\ 0 < x < 2\pi$$

fourier series of the function f(x) is given by

$$f(x) = \frac{(\pi - x)^2}{4} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - - - - - - - - - (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} dx = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{3}\right]_0^{2\pi} = \frac{\pi^2}{6} - -(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx\, dx$$

$$= \left[\frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \{2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi} = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{n^2} - (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} \sin nx\, dx$$

$$= \left[\frac{1}{4\pi}\left[(\pi - x)^2 \left(-\frac{\cos nx}{n}\right) - \{2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$= \left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right] = 0\,;\, b_n = 0 - - - - - - - (4)$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{(\pi - x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + - - -$$

4. Expand $f(x) = \begin{cases} 1; & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ as a Fourier Series.

Sol:- The Fourier series for the function in $(0,22\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left[\int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 0\, dx\right] = \frac{1}{\pi}(x)_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx = \frac{1}{\pi}\left[\int_0^{\pi}(1)\cos nx.dx + \int_{\pi}^{2\pi}(0)\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi} = 0$$

$$= \frac{1}{\pi}(0) \quad (\because \sin 0 = 0, \sin n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.dx + \int_{\pi}^{2\pi} 0.\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.dx\right] = \frac{1}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - \cos 0) = -\frac{1}{n\pi}[(-1)^n - 1]$$

$$\therefore b_n = \begin{cases} 0 & when\ n\ is\ even \\ \frac{2}{n\pi} & when\ n\ is\ odd \end{cases}$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{2} + \frac{2}{n\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin nx = \frac{1}{2} + \frac{2}{n\pi}\left(sinx + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ..\right)$$

5. Obtain Fourier series expansion of $f(x) = (\pi - x)^2$ **in** $0 < x < 2\pi$ **and deduce the value of** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ----- = \frac{\pi^2}{6}$

Sol:-

Given $f(x) = (\pi - x)^2 \; 0 < x < 2\pi$

fourier series of the function f(x) is given by

$$f(x) = (\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx --------- (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 dx$$

$$\frac{1}{\pi}\int_0^{2\pi}[\pi^2 + x^2 - 2\pi x]\,dx = \frac{2\pi^2}{3} -- (2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 \cos nx\,dx$$

$$= \left[\frac{1}{\pi}\left[(\pi - x)^2\frac{\sin nx}{n} - \{2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{4}{n^2} - (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx\,dx$$

$$=\left[\frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0\,;b_n=0------(4)$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x)=(\pi-x)^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{3}+\frac{4\cos x}{1^2}+\frac{4\cos 2x}{2^2}+\frac{4\cos 3x}{3^2}+---$$

Deduction:-

Putting $x=0$ in the above equation we get

$$f(0)=(\pi-0)^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{3}+\frac{4\cos 0}{1^2}+\frac{4\cos 0}{2^2}+\frac{4\cos 0}{3^2}+---$$

$$\pi^2=\frac{\pi^2}{3}+\frac{4}{1^2}+\frac{4}{2^2}+\frac{4}{3^2}+---$$

$$\pi^2-\frac{\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+---\right]$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+----=\frac{\pi^2}{6}$$

Even and Odd Functions:-

A function $f(x)$ is said to be even *if* $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$

Example:- $x^2, x^4+x^2+1, e^x+e^{-x}$ are even functions

$x^3, x, \sin x, \operatorname{cosec} x$ are odd functions

Note1:-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx=0$ when $f(x)$ is an odd function

$=2\int_0^a f(x)\,dx$ when $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

Case (i):- when $f(x)$ is an even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is also an even function

Hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,.\,dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,.\,dx$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an odd function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = 0$$

$\therefore$ If a function $f(x)$ is even in $(-\pi,\pi)$, its Fourier series expansion contains only cosine terms

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx, n = 0,1,2,-----$

Case 2:- when $f(x)$ is an odd function in $(-\pi,\pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \text{ Since } f(x) \text{ is odd}$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = 0$$

Since $\sin nx$ is an odd function; $f(x)\sin nx$ is an even function

$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx$

$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx$

Thus, if a function $f(x)$ defined in $(-\pi,\pi)$ is odd, its Fourier expansion contains only sine terms

$\therefore f(x) = \sum_{n=1}^{\pi} b_n \sin nx$ Where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx.dx$

Even and Odd Functions:-

Problems:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $[-\pi,\pi]$, hence deduce that

(i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$

Sol. Since $f(-x)=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is an even function

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2=\frac{a_0}{2}\sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots \text{———}(1)$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi} x^2 dx=\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}=\frac{2\pi^2}{3} \quad \ldots\ldots \text{———}(2)$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx=\frac{2}{\pi}\int_0^{\pi} x^2\cos nx.dx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$=\frac{2}{\pi}\left[0+2\pi\frac{\cos n\pi}{n^2}+2.0\right]=\frac{4\cos n\pi}{n^2}=\frac{4}{n^2}(-1)^n$ ---------------(3)

Substituting the values of a_0 and a_n from (2) and (3) in (1) we get

$$x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+---\right)\rightarrow(4)$$

Deduction:- Putting $x=0$ in (4), we get

$$0=\frac{\pi^2}{3}-4\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---\right)\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$$

2. Find the Fourier series to represent the function $f(x)=|\sin x|, -\pi<x<\pi$

Sol: Since $|\sin x|$ is an even function, $b_n=0$ *for all* n

Let $f(x)=|\sin x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\rightarrow(1)$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}(-\cos x)_0^{\pi}$

$$=\frac{-2}{\pi}(-1-1)=\frac{4}{\pi}$$

and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx=\frac{2}{\pi}\int_0^{\pi}\sin x.\cos nx\,dx$

$$=\frac{1}{\pi}\int_0^{\pi}[sin(1+n)x+sin(1-n)x]\,dx$$

$$=\frac{1}{\pi}\left[-\frac{cos(1+n)x}{1+n}-\frac{cos(1-n)x}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=-\frac{1}{\pi}\left[\frac{cos(1+n)\pi}{1+n}+\frac{cos(1-n)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=\frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n}+\frac{(-1)^{n+1}-1}{1-n}\right] \quad =\frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}-\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$

$$=\frac{-1}{\pi}\left[(-1)^{n+1}\frac{2}{1-n^2}-\frac{2}{1-n^2}\right]=\frac{2}{\pi(n^2-1)}[(-1)^{n+1}-1]$$

$$=\frac{-2}{\pi(n^2-1)}[1+(-1)^n] \quad (n\neq 1)$$

$$\therefore a_n=\begin{cases}0 & if\ n\ is\ odd\ and\ n\neq 1\\ \frac{-4}{\pi(n^2-1)} & if\ n\ is\ even\end{cases}$$

For $n=1, a_1=\frac{2}{\pi}\int_0^{\pi}\sin x.cons\ x\,dx \quad =\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$

$$=\frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi} \quad =\frac{-1}{2\pi}(\cos 2\pi-1)=0$$

Substituting the values of a_0, a_1 and a_n in (1) We get $|\sin x|=\frac{2}{\pi}+\sum_{n=2,4,--}^{\infty}\frac{-4}{\pi(n^2-1)}\cos nx$

$$=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4---}^{\infty}\frac{\cos nx}{n^2-1}=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad \text{(Replace n by 2n)}$$

Hence $|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+----\right)$

3. *Show that for* $-\pi<x<\pi, \sin ax=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-......\right]$

(a **is not an integer**)

Sol: - As sin ax is an Odd function. It's Fourier series expansion will consist of sine terms only

$$\therefore \sin ax=\sum_{n=1}^{\infty}b_n\sin nx \qquad \text{------------- (1)}$$

Where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin ax.\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}[\cos(a-n)x-\cos(a+n)x]dx$

$[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)]$

$$b_n=\frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n}-\frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi-\cos\pi\sin n\pi}{a-n}-\frac{\sin a\pi\cos n\pi+\cos a\pi\sin n\pi}{a+n}\right]$$

$$b_n=\frac{1}{\pi}\left[\frac{\sin a\pi.\cos n\pi}{a-n}-\frac{\sin a\pi.\cos n\pi}{a+n}\right][\because \sin n\pi=0]$$

$$=\frac{1}{\pi}\sin a\pi\cos n\pi\left(\frac{1}{a-n}-\frac{1}{a+n}\right)=\frac{1}{\pi}\sin a\pi\,(-1)^n\left(\frac{a+n-a+n}{a^2-n^2}\right)=\frac{(-1)^n 2n}{\pi(a^2-n^2)}\sin a\pi$$

Substituting these values in (1), we get

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(a^2-n^2)}\sin nx = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-a^2)}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\ldots\right]$$

4. **Find the Fourier series to represent the function** $\boldsymbol{f(x) = Sin\, x, -\pi < x < \pi}$.

Sol:- since sin x is an odd function $a_0 = a_n = 0$

Let $f(x) = \sum b_n \sin nx, where$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin x \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(1-n)x - \cos(1+n)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi} (n\neq 1) \quad = 0 (n \neq 1)$$

If n=1 $b_1 = \frac{2}{\pi}\int_0^{\pi} sin^2 x\, dx = \frac{2}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}\, dx = \frac{1}{\pi}\left(x - \frac{Sin2x}{2}\right)_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1 \therefore f(x) = b_1 sinx = sinx$

5. *Show that for* $-\pi < x < \pi$, $\sin k x = \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2} - \frac{2\sin 2x}{2^2-k^2} + \frac{3\sin 3x}{3^2-k^2} - \ldots\ldots\right]$

(k **is not an integer**)

Sol: - As sin kx is an Odd function.

It's Fourier series expansion will consist of sine terms only

$$\therefore \sin k x = \sum_{n=1}^{\infty} b_n \sin nx$$

------------- (1)

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\sin kx.\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}[\cos(k-n)x - \cos(k+n)x]dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$b_n = \frac{1}{\pi}\left[\frac{\sin(k-n)x}{k-n} - \frac{\sin(k+n)x}{k+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin k\pi \cos n\pi - \cos\pi \sin n\pi}{k-n} - \frac{\sin k\pi \cos n\pi + \cos k\pi \sin n\pi}{k+n}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\sin k\pi.\cos n\pi}{k-n} - \frac{\sin k\pi.\cos n\pi}{k+n}\right][\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\sin k\pi \cos n\pi\left(\frac{1}{k-n} - \frac{1}{k+n}\right) = \frac{1}{\pi}\sin k\pi\,(-1)^n\left(\frac{k+n-k+n}{k^2-n^2}\right) = \frac{(-1)^n 2n}{\pi(k^2-n^2)}\sin k\pi$$

Substituting these values in (1), we get

$$\sin kx = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(k^2-n^2)}\sin nx = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-k^2)}\sin nx$$

$$= \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2}-\frac{2\sin 2x}{2^2-k^2}+\frac{3\sin 3x}{3^2-k^2}-......\right]$$

Half –Range Fourier Series:-

1) The sine series:-

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx \text{ where } b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$$

2) The cosine series:-

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad \text{where } a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx \text{ and}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx.dx$$

Note:-

1) Suppose $f(x)=x$ in $[0,\pi]$.It can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$
2) If $f(x)=x^2$ in $[0,\pi]$ can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$.

Half –Range Fourier Series:-

Problems:

1. **Find the half range sine series for . $f(x)=x(\pi-x)$, in $0<x<\pi$**

Deduce that $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----=\frac{\pi^3}{32}$

Sol. The Fourier sine series expansion of $f(x)$ in $(0,\pi)$ is $f(x)=x(\pi-x)=\sum_{n=1}^{\infty}b_n\sin nx$

Where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$; $b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin nx.dx$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx.dx$$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2}{n^3}(1-\cos n\pi)\right]=\frac{4}{n\pi^3}\left(1-(-1)^n\right)$$

$$b_n=\begin{cases}0, \text{ when } n \text{ is even}\\ \frac{8}{\pi n^3}, \text{ when } n \text{ is odd}\end{cases}$$

Hence

$$x(\pi-x)=\sum_{n=1,3,5\ldots}\frac{8}{\pi n^3}\sin nx \quad (or) \quad x(\pi-x)=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+----\right)\rightarrow(1)$$

Deduction:-

Putting $x=\frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x-\frac{\pi}{2}\right)=\frac{8}{\pi}\left(sin\frac{\pi}{2}+\frac{1}{3^3}sin\frac{3\pi}{2}+\frac{1}{5^3}sin\frac{5\pi}{2}+---\right)$$

$$\Rightarrow\frac{\pi^2}{4}=\frac{8}{\pi}\left[1+\frac{1}{3^3}sin\left(\pi+\frac{\pi}{2}\right)+\frac{1}{5^3}sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{7^3}sin\left(3\pi+\frac{\pi}{2}\right)+---\right]$$

(or) $\frac{\pi^2}{32}=1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+---$

2.**Find the half- range sine series for the function** $f(x)=\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} in\ (0,\pi)$

Sol. Let $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$ ——(1)

$Then\ b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx=\frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}.\sin nx.dx$

$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\int_0^{\pi}e^{ax}\sin nx.dx-\int_0^{\pi}e^{-ax}\sin nx.dx\right]$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right]_0^{\pi}-\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx-n\cos nx)\right]_0^{\pi}\right]$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n+\frac{n}{a^2+n^2}+\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n-\frac{n}{a^2+n^2}\right]$$

$$=\frac{2n(-1)^n}{\pi(e^{a\pi}-e^{-a\pi})}\left[\frac{e^{-ax}-e^{ax}}{n^2+a^2}\right]=\frac{2n(-1)^{n+1}}{\pi(n^2+a^2)}--\text{——}(2)$$

Substituting (2) in (1), we get

$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx=\frac{2}{\pi}\left[\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-----\right]$

Fourier series of $f(x)$ defined in $[c,c+2l]$

It can be seen that role played by the functions

$1,\cos x,\cos 2x,\cos 3x,.....\sin x,\sin 2x.........$

In expanding a function $f(x)$ defined in $[c,c+2\pi]$ as a Fourier series, will be played by

$$1,\cos\left(\frac{\pi x}{e}\right),\cos\left(\frac{2\pi x}{e}\right),\cos\left(\frac{3\pi x}{e}\right),.....$$

$$\sin\left(\frac{\pi x}{e}\right),\sin\left(\frac{2\pi x}{e}\right),\sin\left(\frac{3\pi x}{e}\right),.....$$

In expanding a function $f(x)$ defined in $[c,c+2l]$

$$(i)\ \int_c^{c+2l}sin\left(\frac{m\pi x}{l}\right).cos\left(\frac{n\pi x}{l}\right)dx=0$$

$$(ii)\ \int_c^{c+2l} sin\left(\frac{m\pi x}{l}\right).sin\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0, & if\ m \neq n \\ l, & if\ m = n \neq 0 \\ 0, & if\ m = n = 0 \end{cases}$$

$$(iii)\ \int_c^{c+2l} cos\left(\frac{m\pi x}{l}\right).cos\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0, & if\ m \neq n \\ l, & if\ m = n \neq 0 \\ 2l, & if\ m = n = 0 \end{cases}$$

[It can be verified directly that, when m, n are integers]

Fourier series of $f(x)$ defined in$[0, 2l]$:

Let $f(x)$ be defined in $[0,2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\, cos\frac{n\pi x}{l} + b_n\, sin\frac{n\pi x}{l}\right] \to (1)$

Where $a_n = \frac{1}{l}\int_0^{2l} f(x)cos\frac{n\pi x}{l}dx \to (2)$

.and $b_n = \frac{1}{l}\int_0^{2l} f(x)sin\frac{n\pi x}{l}dx \to (3)$

Fourier Series Of $f(x)$ Defined In$[-l, l]$:

Let $f(x)$ be defined in $[-l,l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$where\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Fourier series for even and odd functions in$[-l, l]$:-

Let $f(x)$ be defined in $[-l,l]$. If $f(x)$ is even $f(x)\cos\frac{n\pi x}{l}$ is also even

$$\therefore a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)cos\frac{n\pi x}{l}dx$$

And $f(x)sin\frac{n\pi x}{l}$ $is\ odd$

$$\therefore b_n = \frac{1}{l}\int_{-l}^{l} f(x)sin\frac{n\pi x}{l}dx = 0\forall n$$

Hence if $f(x)$ is defined in $[-l,l]$ and is even its Fourier series expansion is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

If $f(x)$ is defined in $[-l,l]$ and its odd its Fourier series expansion is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\quad where\ b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Note:- In the above discussion if we put $2l=2\pi, l=\pi$ we get the discussion regarding the intervals $[0,2\pi]$ and $[-\pi,\pi]$ as special cases

Fourier series of $f(x)$ defined in $[c, c+2l]$

Problems:-

1.Express $f(x)=x^2$ as a Fourier series in $[-l,l]$

Sol: Since $f(-x)=(-x)^2=x^2=f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\ where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Hence $a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l^2}{3}$

$$Also\ a_n=\frac{2}{l}\int_0^l f(x)cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\frac{cos\frac{n\pi x}{1}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

(Since the first and last terms vanish at both upper and lower limits)

$$\therefore a_n = \frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} = \frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - - - - -\right]$$

2. **Obtain Fourier series for f(x) = x^3 in [-1, 1].**

Sol: The given function is x^3 which is odd

a_0=, a_n=0, b_n= $b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^1 x^3 \sin n\pi x\, dx$

$$=2\left[-x^3\frac{\cos n\pi x}{n\pi} + 3x^2\sin\frac{n\pi x}{n^2\pi^2} + 6x\frac{\cos n\pi x}{n^3\pi^3} - 6\sin\frac{n\pi x}{n^4\pi^4}\right]_0^1$$

$$= 2\left[\frac{-(-1)^n}{n\pi} + \frac{6(-1)^n}{n^3\pi^3}\right]$$

$\therefore$ f(x) $=2[\left(\frac{1}{\pi} - \frac{6}{\pi^3}\right)\sin x + \left(-\frac{1}{2\pi} + \frac{6}{2^3\pi^3}\right)\sin 2\pi x + \left(\frac{1}{3\pi} - \frac{6}{3^3\pi^3}\right)\sin 3\pi x + \left(-\frac{1}{4\pi} + \frac{6}{4^3\pi^3}\right)\sin 4\pi x]$

3. Find a Fourier series with period 3 to represent $f(x) = x + x^2 \ in\ (0,3)$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \to (1)$

Here $2l = 3,\quad l = 3/2$ Hence (1) becomes

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \to (2)$$

$$Where\ a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$$

$$and\ a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{4n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

Finally $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx \quad = \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx = \frac{-12}{n\pi}$

Substituting the values of a's and b's in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half- Range Expansion of $f(x)$ in $[0, l]$:-

Some times we will be interested in finding the expansion of $f(x)$ defined in $[0,l]$ in terms of sines only (or) in terms of cosines only. Suppose we want the expansion of $f(x)$ in terms of sine series only. Define $f_1(x) = f(x)$ in $[0,l]$ and $f_1(x) = -f_1(x) \forall n$ with $f_1[2l + x] = f_1(x)$, $f_1(x)$ is an odd function in $[-l,l]$. Hence its Fourier series expansion is given by

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} dx$$

$$where\ b_n = \frac{2}{l}\int_0^1 f_1(x)dx$$

The above expansion is valid for x in $[-l,l]$ in particular for x in $[0,l]$,

$$f_1(x) = f(x) \text{ and } f_1(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} dx \text{ where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l} dx$$

This expansion is called the half- range sine series expansion of $f(x)$ in $[0,l]$. If we want the half – range expansion of $f(x)$ in $[0,l]$, only in terms of cosines, define $f_1(x) = f(x)$ in $[0,l]$ and $f_1(-x) = f_1(x)$ for all x with

$f_1(x + 2l) = f_1(x)$.

Then $f_1(x)$ is even in $[-l,l]$ and hence its Fourier series expansion is given by

$$f_1(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$$

$$where\ a_n = \frac{2}{l}\int_0^l f_1(x)\cos\frac{n\pi x}{l} dx$$

The expansion is valid in $[-l,l]$ and hence in particular on $[0,l]$,

$f_1(x) = f(x)$ hence in $[0,l]$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Where $a_n = \dfrac{2}{l}\displaystyle\int_0^1 f(x)\cos\frac{n\pi x}{l} dx$

1. The half range sine series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$ is given by

   Where $b_n = \dfrac{2}{l}\displaystyle\int_0^1 f(x)\sin\frac{n\pi x}{l} dx$

2. The half range cosine series expansion of $f(x)$ in $[0,l]$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ b_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Problems:-

1. Find the half- range sine series of $f(x)=1\ in\ [0,l]$

Sol: The Fourier sine series of $f(x)$ $in\ [0,l]$ is given by $f(x)=1=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$Here\ b_n = \frac{2}{1}\int_0^l f(x)sin\frac{n\pi x}{1}dx\ = \frac{2}{1}\int_0^l 1.sin\frac{n\pi x}{1}dx$$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l \quad = \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2}{n\pi}(-\cos n\pi+1) = \frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$$\therefore b_n = \begin{cases} 0 & \text{when n is even} \\ \frac{4}{n\pi} & \text{when n is odd} \end{cases}$$

Hence the required Fourier series is $f(x) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{l}$

$$i.e\ 1 = \frac{4}{\pi}\left(\sin\frac{n\pi}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + {}^{1}/_{5}\sin\frac{5\pi x}{l}\ \dots\dots\right)$$

2. Find the half – range cosine series expansion of $f(x)=\sin\left(\frac{\pi x}{l}\right)$ **in the range** $0<x<l$

Sol : The half-range Fourier Cosine Series is given by

$$f(x) = sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\ \dots\dots \quad \text{——}(1)$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l sin\frac{\pi x}{l}dx = \frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l = \frac{-2}{\pi}(\cos\pi - 1) = \frac{4}{\pi}$

and $a_n = \frac{2}{l}\int_0^l f(x)cos\frac{n\pi x}{l}dx\ = \frac{2}{l}\int_0^l sin\left(\frac{\pi x}{l}\right)cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{1}{l}\int_0^l \left[\frac{sin(n+1)\pi x}{l} - \frac{sin(n-1)\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{-\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l} + \frac{\cos(n-1)\pi x/l}{(n+1)\pi/l}\right]_0^l\ (n \neq 1)$$

$$= \frac{1}{l}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]\quad (n \neq 1)$$

When n is odd $a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

When n is even $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$= \frac{-4}{\pi(n+1)(n-1)} \quad (n \neq 1)$

If $n = 1, a_1 = \frac{1}{l}\int_0^l 2\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)dx = \frac{1}{l}\int_0^l \sin\left(\frac{2\pi x}{l}\right)dx$

$= \frac{1}{l}\cdot\frac{1}{2\pi}\left[-cos\left(\frac{2\pi x}{l}\right)\right]_0^l = \frac{-1}{2\pi}(\cos 2\pi - cos0) = {-1}/{2\pi}(1-1) = 0$

from equation(1) we have∴ $\sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3} + \frac{\cos(4\pi x/l)}{3.5} + ---\right]$

3. Obtain the half range cosine series for f(x) = x – x², 0≤ x ≤ 1.

Sol:The half range cosine series for f(x) in 0≤ x ≤ 1 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

Where $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$a_n = 2\int_0^1 (x - x^2)\cos n\pi x\,dx$

$= 2\left[(x - x^2)\frac{\cos n\pi x}{n\pi} + (1-2x)\frac{\cos n\pi x}{n\pi^2}\right]_0^1 = 2\left[(-1)\frac{\cos n\pi x}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = 2\left[\frac{(-1)^{n+1} - 1}{n^2\pi^2}\right]$

∴ The cosine series of f(x) is given by,

$f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1} - 1}{n^2}\right\}\cos n\pi x \quad = \frac{1}{6} - \frac{4}{\pi^2}\left\{\frac{\cos 2\pi x}{2^2} + \frac{\cos 4\pi x}{4^2} + ----\right\}$

4. Obtain the half range sine series for eˣ in 0<x<1.

Sol: The sine series is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Where $bn = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx$

$= 2\int_0^1 e^x \sin n\pi x\,dx = \left[\frac{2e^x}{(1+n^2\pi^2)}[\sin n\pi x - n\pi x \cos n\pi x]\right]_0^1$

$= \frac{2}{(1+n^2\pi^2)}[-n\pi e\cos n\pi + n\pi] = \frac{2}{(1+n^2\pi^2)}[1 - e(-1)^n]$

$\therefore e^x = 2\pi\left[\frac{(1+e)}{1+\pi^2}\sin\pi x + \frac{2(1-e)}{1+4\pi}\sin 2\pi x + \frac{3(1+e)}{1+9\pi}\sin 3\pi x + ---\right]$

UNIT-III

FOURIER TRANSFORMS

Dirichlet's condition :

A function f(x) is said to satisfy dirichlets conditions in the interval (a,b) if

(i) f(x) defined and is single valued function except possibly at a finite number of points in the interval (a,b) and

(ii) f(x) and $f^1(x)$ are piecewise continuous in(a,b)

Fourier Integral Theorem:

If f (x) is defined in $(-l,l)$ and satisfies Dirichlet's condition, then

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt.d\lambda$$

Fourier Sine Integral:

The Fourier sine integral for f(x) is given by

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t dt.d\lambda$$

Fourier Cosine Integral:

The Fourier cosine integral for f(x) is given by

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos\lambda x\int_0^{\infty} f(t).\cos\lambda t.dtd\lambda$$

1. Using Fourier integral show that

$$e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^{\infty}\frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda, a>0, b>0$$

Sol. Since the integral on R.H.S contains sine term use Fourier sine integral formula.

We know that the F.S.I for f(x) is given by.

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t.dt.d\lambda \ldots\ldots\ldots\ldots\ldots.. (1)$$

Here $f(x)=e^{-ax}-e^{-bx}; \quad \therefore f(t)=e^{-at}-e^{-bt}$

$$\therefore e^{-ax}-e^{-bx}=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty\left(e^{-at}-e^{-bt}\right)\sin\lambda t\,dt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty e^{-at}\sin\lambda t\,dt-\int_0^\infty e^{-bt}\sin\lambda tdt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{e^{-at}}{a^2+\lambda^2}(-a\sin\lambda t-\lambda\cos\lambda t)-\frac{e^{-bt}}{b^2+\lambda^2}(-b\sin\lambda t-\lambda\cos\lambda t)\right]_0^\infty d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\lambda.\left[\frac{1}{\lambda^2+a^2}-\frac{1}{\lambda^2+b^2}\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\frac{\lambda\left(b^2-a^2\right)}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

$$\therefore e^{-ax}-e^{-bx}=\frac{2\left(b^2-a^2\right)}{\pi}\int_0^\infty \frac{\lambda.\sin\lambda x}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

Hence proved

2. Using Fourier Integral, show that $\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}.\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2}\, if\ 0<x<\pi\\ 0, if\ x>\pi\end{cases}$

Sol. Since the integral on R.H.S. contains the sine term we use Fourier Sine Integral formula.

The Fourier Sine Integral for f(x) is given by.

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t).\sin\lambda t\,dt.d\lambda \text{ --------------------- (1)}$$

Let $f(x)=\begin{cases}\frac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$ ----------------------------(2)

Using (2) in (1), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\int_0^\pi f(t)\sin\lambda t\,dt+\int_\pi^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\int_0^\pi \frac{\pi}{2}.\sin\lambda t\,dt\right]d\lambda \quad =\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\frac{\pi}{2}\left(\frac{-1}{\lambda}\right).\cos\lambda t\right]_0^\pi d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{-\pi}{2\lambda}(\cos\lambda\pi-1)\right]d\lambda=\frac{\pi}{2}\times\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{1-\cos\lambda\pi}{\lambda}\right]d\lambda$$

$$\therefore f(x)=\int_0^\infty \frac{(1-\cos\lambda\pi)}{\lambda}\sin\lambda x\,d\lambda \ or\ \int_0^\infty \frac{(1-\cos\lambda\pi)}{\lambda}.\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$$

3. Express $f(x) = \begin{cases} 1 \text{ for } 0 \le x \le \pi \\ 0 \text{ for } \quad x > \pi \end{cases}$ as a Fourier cosine integral and hence evaluate $\int_0^\infty \frac{\cos \lambda x \sin \lambda\pi}{\lambda} d\lambda$

Sol:- Fourier cosine integral of f(x) is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda$$

$$f(x) = \begin{cases} 1 \text{ for } 0 \le x \le \pi \\ 0 \text{ for } \quad x > \pi \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\pi \cos\lambda t\, dt\right]d\lambda = \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\frac{\sin\lambda t}{\lambda}\right]_0^\pi d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x \sin\lambda\pi}{\lambda}\, d\lambda$$

$$\therefore \int_0^\infty \frac{\cos\lambda x\sin\lambda\pi}{\lambda} d\lambda = \frac{\pi}{2}\, f(x) = \begin{cases} \frac{\pi}{2} & \text{for } \quad 0 \le x \le \pi \\ 0 & \text{for } \quad x > \pi \end{cases}$$

At $x = \pi$ which is a point of discontinuity of f(x) ,the value of the above integral

$$\int_0^\infty \frac{\cos\lambda x\sin\lambda\pi}{\lambda} d\lambda = \frac{\pi}{2}\left(\frac{1+0}{2}\right) = \frac{\pi}{4}$$

<u>FOURIER TRANSFORM OR COMPLEX FOURIER TRANSFORM</u>

<u>The Infinite Fourier Transform of f(x)</u> :

The Fourier transform of a function f(x) is given by.

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x).e^{ipx}\, dx$$

The inverse Fourier transform of F(p) is given by.

$$f(x) = F^{-1}\{F(p)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p).e^{-ipx}\, dp$$

<u>Fourier sine Transform</u>:

The Fourier sine Transform of a function f(x) is given by

$$F_s\{f(x)\} = F_s(p) = \int_0^\infty f(x).\sin px\, dx$$

The inverse Fourier sine Transform of F_s (p) is given by

$$f(x) = F_s^{-1}\{F_s(p)\} = \frac{2}{\pi}\int_0^\infty F_s(p).\sin px\, dp$$

Fourier cosine Transform:

The Fourier cosine Transform of a function f(x) is given by

$$F_c\left\{f(x)\right\}=F_c(p)=\int_0^{\infty} f(x).\cos px\,dx$$

The inverse Fourier cosine Transform of $F_c(p)$ is given by

$$f(x)=F_c^{-1}\left\{F_c(p)\right\}=\frac{2}{\pi}\int_0^{\infty} F_c(p).\cos px\,dp$$

Problems:

1. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} x^2, & |x|<a \\ 0, & |x|>a \end{cases}$

Sol. We have $F\left\{f(x)\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a} e^{ipx} f(x)dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} e^{ipx} f(x)\,dx+\int_{a}^{\infty} e^{ipx} f(x)\,dx\right]$$

$$F\left\{f(x)\right\}=\frac{1}{\sqrt{2\pi}}\left[x^2\frac{e^{ipx}}{ip}+\frac{2}{p^2}xe^{ipx}+\frac{2i}{p^3}e^{ipx}\right]_{-a}^{a}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{ip}\left(e^{ipa}-e^{-ipa}\right)+\frac{2a}{p^2}\left(e^{ipa}+e^{-ipa}\right)+\frac{2i}{p^3}\left(e^{ipa}-e^{-ipa}\right)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2a^2\sin ap}{p}+\frac{4a}{p^2}\cos ax-\frac{4}{p^3}\sin ap\right]$$

2. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$ *and hence evaluate*

$$\int_0^{\infty}\frac{\sin p}{p}dp.and\int_{-\infty}^{\infty}\frac{\sin ap.\cos px}{p}dp$$

Sol. We have $F\left\{f(x)\right\}=\int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$=\int_{-\infty}^{-a} e^{ipx} f(x)dx+\int_{-a}^{a} e^{ipx} f(x)\,dx+\int_{a}^{\infty} e^{ipx} f(x)\,dx=\int_{-a}^{+a}(1)e^{ipx}dx$$

$$= \left[\frac{e^{ipx}}{ip}\right]_{-a}^{+a} = \frac{e^{ipa} - e^{-ipa}}{ip} = \frac{2}{p}.\frac{e^{ipa} - e^{-ipa}}{2i} = \frac{2\sin pa}{p} = F\{f(x)\} = \frac{2\sin pa}{p} = F(p)$$

We know that $F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x) dx$

Then by the inversion formula,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.F(p)\,dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.\frac{2\sin pa}{p}\,dp$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{2\sin ap}{p}.(\cos px - i\sin px)\,dp\right] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap}{p}\cos px\,dp - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap}{p}\sin px\,dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp$$ [Since the second integral is an odd]

$$or\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

If x=0 and a=1, then

$$\int_{-\infty}^{\infty}\frac{\sin p}{p}dp = \pi \ or\ 2\int_{0}^{\infty}\frac{\sin p}{p}dp = \pi \ or\ \int_{0}^{\infty}\frac{\sin p}{p}dp = \frac{\pi}{2}$$

Note: $\int_{0}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$

3. Find the Fourier transform of f(x) defined by $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate $(i)\ \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx \quad (ii)\ \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}dx$

Sol. We have $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x)dx = \int_{-\infty}^{-1} e^{ipx} f(x)dx + \int_{-1}^{1} e^{ipx} f(x)dx + \int_{1}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-1}^{1}(1-x^2).e^{ipx}dx = \left\{\left[\frac{(1-x^2)}{ip} - \frac{(-2x)}{i^2p^2} + \frac{(-2)}{i^3p^3}\right]e^{ipx}\right\}_{x=-1}^{1} = \left(\frac{-2}{p^2} + \frac{2}{ip^3}\right)e^{ip} - \left(\frac{2}{p^2} + \frac{2}{ip^3}\right)e^{-ip}$$

$$= \frac{-2}{p^2}\left(e^{ip} + e^{-ip}\right) + \frac{2}{ip^3}\left(e^{ip} - e^{-ip}\right) = \frac{-4}{p^2}\left(\frac{e^{ip} + e^{-ip}}{2}\right) + \frac{4i}{ip^3}\left(\frac{e^{ip} - e^{-ip}}{2i}\right) = \frac{-4}{p^2}\cos p + \frac{4}{p^3}\sin p$$

$$=\frac{4}{p^3}(\sin p - p\cos p) = F(p)$$

Second Part: By inversion formula, we have

i. $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.F(p)dp$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.\frac{4(\sin p - p\cos p)}{p^3}dp = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Putting $x=\frac{1}{2}\, in\, (1), we\, get$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i p/2}.\frac{4(\sin p - p\cos p)}{p^3}dp = \begin{cases} 1-\frac{1}{4}=\frac{3}{4} \\ 0 \end{cases}$$

$$or \int_{-\infty}^{\infty}\frac{1}{p^3}(p\cos p - \sin p)e^{-i p/2}dp = \frac{-3\pi}{8}$$

$$or \int_{-\infty}^{\infty}\frac{1}{p^3}(p\cos p - \sin p)\left(\cos p/2 - i\sin p/2\right)dp = \frac{-3\pi}{8}$$

$$or \int_{-\infty}^{\infty}\frac{p\cos p - \sin p}{p^3}.\cos p/2\, dp = \frac{-3\pi}{8} \left(Equating\ real\ parts\right)$$

$$or\ 2\int_{0}^{\infty}\frac{p\cos p - \sin p}{p^3}.\cos p/2\, dp = \frac{-3\pi}{8} \text{ [since integral is even]}$$

$$or \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx = \frac{-3\pi}{16}$$

ii. Putting $x=0\, in\, (1), we\, get$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{p^3}(\sin p - p\cos p)dp = 1 \quad or \int_{-\infty}^{\infty}\frac{\sin p - p\cos p}{p^3}dp = \frac{\pi}{2}$$

$$or\ 2\int_{0}^{\infty}\frac{\sin p - p\cos p}{p^3}dp = \frac{\pi}{2} [\because Integral\ is\ even] \quad or \int_{0}^{\infty}\frac{p\cos p - \sin p}{p^3}dp = -\frac{\pi}{4}$$

$$or \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}dx = -\frac{\pi}{4}$$

4. **Find the Fourier Transform of f(x)=** $\begin{cases} 0 & \text{if } x \le a \\ 1 & \text{if } a < x \le b \\ 0 & \text{if } x \ge b \end{cases}$

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{ipx}dx = \frac{1}{\sqrt{2\pi}}\int_a^b e^{ipx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipx}}{ip}\right]_a^b$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ibx} - e^{iax}}{ip}\right]$$

5. Find the Fourier Transform of f(x) defined by $f(x) = e^{\frac{-x^2}{2}}, -\infty < x < \infty$ **or,**

Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ **is reciprocal.**

Sol. We have $F\{f(x)\} = \int_{-\infty}^{\infty} f(x).e^{ipx}dx = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}.e^{ipx}dx$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2} e^{-p^2/2} dx = e^{-p^2/2}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2}dx$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t \text{ so that } \frac{1}{2}(x-ip)^2 = t^2 \text{ and } dx = \sqrt{2}dt$

$$\therefore F\{f(x)\} = e^{-p^2/2}\int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}dt = \sqrt{2}.e^{-p^2/2}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$= \sqrt{2}.e^{-p^2/2}\sqrt{\pi} \left[\because \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}\right] = \sqrt{2\pi}.e^{-p^2/2}$$

6. Find the Fourier Transform of f(x) defined by

$$f(x) = \begin{cases} e^{iqx}, & \alpha < x < \beta \\ 0, & x < \alpha \text{ and } x > \beta \end{cases} \text{ or } f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$$

Sol. We have $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-\infty}^{\alpha} e^{ipx} f(x)dx + \int_{\alpha}^{\beta} e^{ipx} f(x)dx + \int_{\beta}^{\infty} e^{ipx} f(x)dx$$

$$= \int_{\alpha}^{\beta} e^{ipx}.e^{iqx} dx = \int_{\alpha}^{\beta} e^{i(p+q)x} dx = \frac{1}{i(p+q)}\left[e^{i(p+q)x}\right]_{\alpha}^{\beta} = \frac{e^{i(p+q)\alpha} - e^{i(p+q)\beta}}{i(p+q)} = F(p)$$

The finite Fourier sine and cosine Transforms:

- The finite Fourier sine transform of f(x) when 0<x<l, is defined as

$$F_s\{f(x)\} = \int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx = F_s(n)$$

Where n is an integer.
The inverse Fourier sine transform of F_s (n) is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n).\sin\frac{n\pi x}{l}$$

- The finite Fourier cosine transform of f(x), when 0<x<l, is given by

$$F_c\{f(x)\} = \int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx = F_c(n)$$

Where n is an integer.
The inverse Fourier sine transform of F_c (n) is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_c(n).\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n).\cos\left(\frac{n\pi x}{l}\right)$$

PROBLEMS RELATED TO INFINITE FOURIER SINE AND COSINE TRANSFORMS:

1. **Find the Fourier cosine transform of the function f(x) defined by** $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Sol. We have $F_c\{f(x)\} = \int_0^{\infty} f(x).\cos px\,dx = \int_0^a f(x)\cos px\,dx + \int_a^{\infty} f(x)\cos px\,dx$

$$= \int_0^a \cos x.\cos px\,dx = \frac{1}{2}\int_0^a 2\cos px.\cos x dx = \frac{1}{2}\int_0^a \left[\cos(p-1)x + \cos(p+1)x\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{(p-1)}\sin(p-1)x + \frac{1}{(p+1)}\sin(p+1)x\right]_0^a = \frac{1}{2}\left[\frac{\sin(p-1)a}{(p-1)} + \frac{\sin(p+1)a}{p+1}\right]$$

2. **Find the Fourier sine transform of f(x) defined by** $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Sol. We have $F_s\{f(x)\} = \int_0^\infty f(x).\sin px\,dx$

$$= \int_0^a f(x).\sin px\,dx + \int_a^\infty f(x)\sin px\,dx = \int_0^a \sin x \sin px dx = \frac{1}{2}\int_0^a 2\sin x.\sin px\,dx$$

$$= \frac{1}{2}\int_0^a \left[\cos(1-p)x - \cos(1+p)x\right]dx = \frac{1}{2}\left[\frac{\sin(1-p)x}{1-p} - \frac{\sin(1+p)x}{1+p}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(1-p)a}{1-p} - \frac{\sin(1+p)a}{1+p}\right]$$

3. Find the Fourier cosine transform of $2e^{-3x} + 3e^{-2x}$

We have $F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^\infty \left(2e^{-3x} + 3e^{-2x}\right)\cos px\,dx$

$$= 2\int_0^\infty e^{-3x} \cos px\,dx + 3\int_0^\infty e^{-2x}\cos px\,dx$$

$$= 2\left[\frac{e^{-3x}}{9+p^2}(-3\cos px + p\sin px)\right]_0^\infty + 3\left[\frac{e^{-2x}}{4+p^2}(-2\cos px + p\sin px)\right]_0^\infty$$

$$= -2\times\frac{1}{9+p^2}\times(-3) - 3\frac{1}{4+p^2}\times(-2) = \frac{6}{p^2+25} + \frac{6}{p^2+4}$$

4. Find Fourier cosine and sine transforms of $e^{-ax}, a > 0$ **and hence deduce the inversion formula (or) deduce the integrals** $i.\ \int_0^\infty \frac{\cos px}{a^2+p^2}dp \quad ii.\ \int_0^\infty \frac{p\sin px}{a^2+p^2}dp$

Sol. Let $f(x) = e^{-ax}$

We have $F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$

$$= \int_0^\infty e^{-ax}.\cos px\,dx = \left[\frac{e^{-ax}}{a^2+p^2}(-a\cos px + p\sin px)\right]_0^\infty$$

$$= -\frac{1}{a^2+p^2}\left(-a(1) + p(0)\right) = \frac{a}{a^2+p^2} = F_c(p) \text{ and } F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^\infty e^{-ax}.\sin px\,dx = \left[\frac{e^{-ax}}{a^2+p^2}(-a\sin px - p\cos px)\right]_0^\infty = -\frac{1}{a^2+p^2}\left(-a(0) - p(1)\right) = \frac{p}{a^2+p^2} = F_s(p)$$

Deduction: i. Now by the inverse Fourier cosine Transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp$$

$$\therefore e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\,dp=\frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}dp \text{ or } \int_0^\infty \frac{\cos px}{a^2+p^2}dp=\frac{\pi}{2a}e^{-ax}$$

ii. Now by the inverse Fourier sine transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(p).\sin px\,dp. \quad \therefore e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

$$\text{or } \int_0^\infty \frac{p\sin px}{a^2+p^2}dp=\frac{\pi}{2}e^{-ax}$$

5. Find the Fourier sine and cosine transform of $2e^{-5x}+5e^{-2x}$

Sol. Let $f(x)=2e^{-5x}+5e^{-2x}$

i. The Fourier sine transform of f(x) is given by

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx=\int_0^\infty\left(2e^{-5x}+5e^{-2x}\right)\sin px\,dx$$

$$=2.\int_0^\infty e^{-5x}.\sin px\,dx+5\int_0^\infty e^{-2x}.\sin px\,dx$$

$$=2.\left[\frac{e^{-5x}}{25+p^2}(-5\sin px-p\cos px)\right]_0^\infty+5\left[\frac{e^{-2x}}{4+p^2}(-2\sin px-p\cos px)\right]_0^\infty$$

$$=-2.\times\frac{1}{25+p^2}(-p)-5\times\frac{1}{4+p^2}(-p)=\frac{2p}{p^2+25}+\frac{5p}{p^2+4}$$

ii. We have $F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx=\int_0^\infty\left(2e^{-5x}+5e^{-2x}\right)\cos px\,dx$

$$=2\int_0^\infty e^{-5x}\cos px\,dx+5\int_0^\infty e^{-2x}\cos px\,dx$$

$$=2\left[\frac{e^{-5x}}{25+p^2}(-5\cos px+p\sin px)\right]_0^\infty+5\left[\frac{e^{-2x}}{4+p^2}(-2\cos px+p\sin px)\right]_0^\infty$$

$$=-2\times\frac{1}{25+p^2}\times(-5)-5\frac{1}{4+p^2}\times(-2)=\frac{10}{p^2+25}+\frac{10}{p^2+4}$$

6. Find the Fourier sine Transform of $e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}dx$

Sol. Let $f(x)=e^{-|x|}$

We have

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx=\int_0^\infty e^{-|x|}\sin px\,dx$$

$$=\int_0^\infty e^{-x}\sin px\,dx \quad [\because |x|=x \text{ in } (0,\infty)]$$

$$=\left[\frac{e^{-x}}{1+p^2}(-\sin px-p\cos px)\right]_0^\infty=-\frac{1}{1+p^2}(-p)=\frac{p}{1+p^2}=F_s(p)$$

Now by the inverse Fourier sine transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp \;\therefore e^{-|x|}=\frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2}\sin px\,dp$$

Chang x to m on both sides

$$e^{-|m|}=\frac{2}{\pi}\int_0^\infty \frac{p\sin pm}{1+p^2}dp=\frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}dx, \text{where } p \text{ is replaced by } x$$

$$\therefore \int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-|m|}$$

7. Show that the Fourier sine transform of

$$f(x)=\begin{cases} x, & \text{for } 0<x<1\\ 2-x, & \text{for } 1<x<2\\ 0, & \text{for } x>2\end{cases} \quad is \quad \frac{2\sin p.(1-\cos p)}{p^2}$$

Sol. By definition, $F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx$

$$=\int_0^1 f(x).\sin px\,dx+\int_1^2 f(x)\sin px\,dx+\int_2^\infty f(x)\sin px\,dx$$

$$=\int_0^1 x.\sin px\,dx+\int_1^2 (2-x).\sin px\,dx$$

$$=\left[-\frac{x}{p}\cos px+\frac{1}{p^2}\sin px\right]_0^1+\left[\frac{-(2-x)}{p}\cos px+\frac{(-1)}{p^2}\sin px\right]_1^2$$

$$=\frac{-\cos p}{p}+\frac{1}{p^2}\sin p-\frac{1}{p^2}\sin 2p+\frac{\cos p}{p}+\frac{1}{p^2}\sin p$$

$$=\frac{2\sin p-\sin 2p}{p^2}=\frac{2\sin p-2\sin p\cos p}{p^2}=\frac{2\sin p(1-\cos p)}{p^2}$$

8. Find the Fourier cosine transform of $f(x)$ defined by $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

By definition, $F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx$

$$=\int_0^1 f(x).\cos px\,dx+\int_1^2 f(x)\cos px\,dx+\int_2^\infty f(x)\cos px\,dx$$

$$=\int_0^1 x.\cos px\,dx+\int_1^2 (2-x).\cos px\,dx$$

$$=\left[\frac{x}{p}\sin px+\frac{1}{p^2}\cos px\right]_0^1+\left[\frac{(2-x)}{p}\sin px+\frac{(-1)}{p^2}\cos px\right]_1^2$$

$$=\frac{\sin p}{p}+\frac{\cos p}{p^2}-\frac{1}{p^2}\cos 2p+\frac{\sin p}{p}+\frac{1}{p^2}\cos p=2\frac{\sin p}{p}+2\frac{\cos p}{p^2}-\frac{1}{p^2}\cos 2p=\frac{2p^2\sin p+2\cos p-\cos 2p}{p^2}$$

9. Find the inverse Fourier cosine transform $f(x)$ of

$$F_c(p)=\begin{cases}\frac{1}{2a}\left(a-\frac{p}{2}\right), & when \quad p<2a \\ 0, & when \quad p\geq 2a\end{cases}$$

Sol. From the inverse Fourier cosine transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp$$

$$=\frac{2}{\pi}\left[\int_0^{2a}\frac{1}{2a}\left(a-\frac{p}{2}\right)\cos px\,dp+\int_{2a}^\infty 0.\cos px\,dp\right]$$

$$=\frac{2}{\pi}\times\frac{1}{2a}\int_0^{2a}\left(a-\frac{p}{2}\right)\cos px\,dp=\frac{1}{a\pi}.\left[\frac{a-p/2}{x}.\sin px-\frac{1}{2x^2}\cos px\right]_{p=0}^{2a}$$

$$=\frac{1}{a\pi}\left[0-\frac{1}{2x^2}\cos 2ax+\frac{1}{2x^2}\right]=\frac{1}{2a\pi x^2}(1-\cos 2ax)=2\frac{\sin^2 ax}{2a\pi x^2}=\frac{\sin^2 ax}{a\pi x^2}$$

10. Find the Fourier cosine transform of $(a)\quad e^{-ax}\cos ax \quad (b)\quad e^{-ax}\sin ax$

Sol. (a). Let $f(x)=e^{-ax}\cos ax.$ *Then*

$$F_c\{f(x)\}=\int_0^\infty f(x).\cos px\,dx$$

$$=\int_0^\infty a^{-ax}\cos ax\cos px\,dx=\frac{1}{2}\int_0^\infty e^{-ax}.2\cos px.\cos ax\,dx$$

$$=\frac{1}{2}\int_0^\infty e^{-ax}\left[\cos(p+a)x+\cos(p-a)x\right]dx$$

$$=\frac{1}{2}\left[\int_0^\infty e^{-ax}\cos(p+a)x+\int_0^\infty e^{-ax}\cos(p-a)x\,dx\right]$$

$$=\frac{1}{2}\left[\left[\frac{e^{-ax}}{a^2+(p+a)^2}\left(-a\cos(p+a)x+(p+a)\sin(p+a)x\right)\right]_0^\infty+\left[\frac{e^{-ax}}{a^2+(p-a)^2}\left(-a\cos(p-a)x+(p-a)\sin(p-a)x\right)\right]_0^\infty\right]$$

$$=\frac{1}{2}\left[-\frac{1}{a^2+(p+a)^2}(-a.1)-\frac{1}{a^2+(p-a)^2}(-a)1\right]$$

$$=\frac{1}{2}\left[\frac{a}{a^2+(p+a)^2}+\frac{a}{a^2+(p-a)^2}\right]=\frac{a}{2}\left[\frac{a^2+(p-a)^2+a^2+(p+a)^2}{\left[a^2+(p+a)^2\right].\left[a^2+(p-a)^2\right]}\right]$$

$$=\frac{a}{2}\times\frac{2a^2+2(a^2+p^2)}{\left[a^2+(p+a)^2\right].\left[a^2+(p-a)^2\right]}=\frac{a(2a^2+p^2)}{\left(a^2+(p+a)^2\right).\left(a^2+(p-a)^2\right)}$$

b. Let $f(x)=e^{-ax}\sin ax$ *Then*

$$F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx$$

$$=\int_0^\infty e^{-ax}.\sin ax\cos px\,dx=\frac{1}{2}\int_0^\infty e^{-ax}(2\cos px\sin ax)dx$$

$$=\frac{1}{2}\int_0^\infty e^{-ax}\left[\sin(p+a)x-\sin(p-a)x\right]dx$$

$$=\frac{1}{2}\left\{\left[\frac{e^{-ax}}{a^2+(p+a)^2}\left(-a\sin(p+a)x-(p+a)\cos(p+a)x\right)\right]_0^\infty-\left[\frac{e^{-ax}}{a^2+(p-a)^2}\left(-a\sin(p-a)x-(p-a)\cos(p-a)x\right)\right]_0^\infty\right\}$$

$$=\frac{1}{2}\left[\frac{p+a}{a^2+(p+a)^2}-\frac{(p-a)}{a^2+(p-a)^2}\right]=\frac{1}{2}\left(\frac{p}{p^2+(p+a)^2}-\frac{p}{p^2+(p-a)^2}\right)+\frac{1}{2}\left(\frac{a}{p^2+(p+a)^2}+\frac{a}{a^2+(p-a)^2}\right)$$

$$=\frac{p}{2}\times\frac{(-4ap)}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}+\frac{a}{2}\times\frac{2\left(p^2+2a^2\right)}{\left(p^2+(p+a)^2\right)\left(p^2+(p-a)^2\right)}$$

$$=\frac{-2ap^2}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}+\frac{a\left(p^2+2a^2\right)}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}$$

Note: (i) $F_s\{x.f(x)\}=-\frac{d}{dp}\{F_c(p)\}$

(ii) $F_c\{x\,f(x)\}=\frac{d}{dp}\{F_s(p)\}$

11. Find the fourier sine transform of $\frac{1}{x}$

Sol:the fourier sine transform of the given function f(x)= $\frac{1}{x}$

$F_s\{f(x)\}=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\sin px\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}.\sin px\,dx$ put px=t dx=$\frac{dt}{p}$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{\frac{t}{p}}.\frac{dt}{p}$

$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}.dt = \sqrt{\frac{2}{\pi}.\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$ $\left(since \int_0^\infty \frac{\sin t}{t}.dt = \frac{\pi}{2}\right)$

12. Find the Fourier sine and cosine transform of xe^{-ax}

Sol. Let $f(x)=e^{-ax}$

Fourier sine Transform:

We know that $F_s\{x\,f(x)\}=\frac{-d}{dp}\{F_c(p)\}=\frac{-d}{dp}\{F_c\{f(x)\}\}$

$$\therefore F_s\{x.e^{-ax}\}=\frac{-d}{dp}\left[F_c\{e^{-ax}\}\right]=\frac{-d}{dp}\left(\frac{a}{p^2+a^2}\right)=(-a)\left(-\frac{1}{(p^2+a^2)^2}\right).2p=\frac{2ap}{(p^2+a^2)^2}$$

Fourier cosine Transform:

We know that $F_c\{x.f(x)\}=\frac{d}{dp}\left[F_s(p)\right]=\frac{d}{dp}\left[F_s\{f(x)\}\right]$

$$\therefore F_c\{x.e^{-ax}\}=\frac{d}{dp}\left[F_s\{e^{-ax}\}\right]=\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)=\frac{(p^2+a^2).1-p.(2p)}{(p^2+a^2)^2}=\frac{a^2-p^2}{(p^2+a^2)^2}$$

13. Find the Fourier sine transform of $\frac{x}{a^2+x^2}$ and Fourier cosine transform of $\frac{1}{a^2+x^2}$

Sol. Fourier sine transforms:

We have $F_s\left\{e^{-ax}\right\} = \frac{p}{a^2+p^2} = F_s(p)$

The inverse Fourier sine transforms of e^{-ax} is $e^{-ax} = \frac{2}{\pi}\int_0^\infty F_s(p).\sin px\,dp = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$

$$or\ \int_0^\infty \frac{p\sin px}{a^2+p^2}dp = \frac{\pi}{2}e^{-ax}$$

Changing p to x and x to p, we get

$$\int_0^\infty \frac{x}{a^2+x^2}\sin xp\,dx = \frac{\pi}{2}e^{-ap}$$

Hence $F_s\left\{\frac{x}{a^2+x^2}\right\} = \frac{\pi}{2}e^{-ap}$

Fourier cosine Transform:

We have $F_c\left\{e^{-ax}\right\} = \frac{a}{p^2+a^2} = F_c(p)$

The inverse Fourier cosine transform of e^{-ax} *is*

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp = \frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\,dp\ or \int_0^\infty \frac{1}{p^2+a^2}\cos px\,dp = \frac{\pi}{2a}e^{-ax}$$

Changing p to x and x to p

$$\int_0^\infty \frac{1}{x^2+a^2}\cos xp.dx = \frac{\pi}{2a}.e^{-ap}$$

Hence $F_c\left\{\frac{1}{x^2+a^2}\right\} = \frac{\pi}{2a}.e^{-ap}$

14. Find the Fourier sine and cosine transform of $f(x) = \frac{e^{-ax}}{x}$ and deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin sx\,dx = Tan^{-1}\left(\frac{s}{a}\right) - Tan^{-1}\left(\frac{s}{b}\right)$$

Sol. **Fourier Sine Transforms:**

We have $F_s\{f(x)\} = \int_0^\infty f(x).\sin px\,dx$

$$= \int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx$$

$$\therefore F_s\{f(x)\} = \int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx$$

Differentiation w.r.t 'p', we get

$$\frac{d}{dp}\left[F_s\{f(x)\}\right] = \frac{d}{dp}\left[\int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx\right]$$

$$= \int_0^\infty \frac{\partial}{\partial p}\left(\frac{e^{-ax}}{x}.\sin px\right)dx = \int_0^\infty \frac{e^{-ax}}{x}.x.\cos px\,dx$$

$$= \int_0^\infty e^{-ax}.\cos px\,dx = \left[\frac{e^{-ax}}{a^2+p^2}(-a\cos px + p\sin px)\right]_0^\infty$$

$$\frac{d}{dp}\left[F_s\{f(x)\}\right] = \frac{a}{p^2+a^2}$$

Integrating w.r.t. p

$$F_s\{f(x)\} = \int_0^\infty \frac{a}{p^2+a^2}dp = Tan^{-1}\left(\frac{p}{a}\right) + c$$

If p=0 then $F_s\{f(x)\} = 0\, and\; c = 0$

$$\therefore F_s\{f(x)\} = Tan^{-1}\left(p/a\right)\ if\ \ p>0$$

$$or\ \ F_s\left\{\frac{e^{-ax}}{x}\right\} = Tan^{-1}\left(p/a\right).if\ \ p>0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Deduction: We know that the Fourier sine transform of f(x) is given by

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx \ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Suppose let $f(x) = \dfrac{e^{-ax} - e^{-bx}}{x}$ ……………….(3)

Using (3) in (2), we get

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin px\,dx = \int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx - \int_0^\infty \frac{e^{-bx}}{x}.\sin px\,dx$$

$$= F_s\left\{\frac{e^{-ax}}{x}\right\} - F_s\left\{\frac{e^{-bx}}{x}\right\} = Tan^{-1}\left(p/a\right) - Tan^{-1}\left(p/b\right)[\text{using}\,(1)]$$

$$or \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin sx\,dx = Tan^{-1}\left(s/a\right) - Tan^{-1}\left(s/b\right)$$

Fourier Cosine Transform:

We have $F_c\left\{e^{-ax}\right\} = \frac{a}{p^2+a^2} = F_c(p)$

The inverse Fourier cosine transform of e^{-ax} *is*

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp = \frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\,dp \quad or \int_0^\infty \frac{1}{p^2+a^2}\cos px\,dp = \frac{\pi}{2a}e^{-ax}$$

Changing p to x and x to p

$$\int_0^\infty \frac{1}{x^2+a^2}\cos xp.dx = \frac{\pi}{2a}.e^{-ap}$$

Hence $F_c\left\{\frac{1}{x^2+a^2}\right\} = \frac{\pi}{2a}.e^{-ap}$

15. Find the finite Fourier sine & cosine transform of f(x), defined by f(x)=2x, where $o < x < 2\pi$

Sol. We have $F_s\{f(x)\} = \int_0^l f(x).\sin\frac{(n\pi x)}{l}dx$

$$= \int_0^{2\pi} 2x.\sin\left(\frac{nx}{2}\right)dx = 2\left[-\frac{2}{n}.x.\cos\frac{nx}{2} + \frac{4}{n^2}\sin\frac{nx}{2}\right]_0^{2\pi} = 2\left[-\frac{4\pi}{n}\cos n\pi + \frac{4}{n^2}\sin n\pi\right] = \frac{-8\pi}{n}\cos n\pi$$

$$= \frac{8\pi}{n}(-1)^{n+1} = F_s(n)$$

$$Also\,F_c\{f(x)\} = \int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^{2\pi} 2x.\cos\left(\frac{nx}{2}\right)dx = 2\left[\frac{2}{n}.x.\sin\frac{nx}{2} + \frac{4}{n^2}\cos\frac{nx}{2}\right]_0^{2\pi}$$

$$= 2\left[\frac{4}{n}\sin n\pi + \frac{4}{n^2}\cos n\pi - \frac{4}{n^2}\right] = 2\times\frac{4}{n^2}(\cos n\pi - 1) = \frac{8}{n^2}\left[(-1)^n - 1\right] = F_c(n)$$

16. Find the finite Fourier sine transform of f(x), defined by f(x) =2x, where $o < x < 4$

Sol:-The finite fourier sine transform of f(x) in 0<x<l

$$F_s\{f(x)\} = \int_0^l f(x).\sin\frac{(n\pi x)}{l}dx$$ Here f(x) = 2x and $l=4$

$$= \int_0^4 2x.\sin\left(\frac{n\pi x}{4}\right)dx = \left[2x\left(-\frac{4}{n\pi}\right)\cos\frac{n\pi x}{4}\right]_0^4 + \int_0^4 2\left(\frac{4}{n\pi}\right)\cos\frac{n\pi x}{4}dx$$

$$= \left[-\frac{8}{n\pi}x\cos\frac{n\pi x}{4}\right]_0^4 + \frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_0^4 = -\frac{32}{n\pi}(\cos n\pi - 0) + 0 = -\frac{32}{n\pi}(-1)^n$$

17. Find the inverse finite sine transform f(x) if $F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2}$ $where\ 0 < x < \pi$

Sol. From the inverse finite sine transform, we have

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty}F_s(n).\sin\left(\frac{n\pi x}{l}\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2\pi^2}\right)\sin nx = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2}\right)\sin nx$$

18. Find the inverse finite cosine transform f(x), if

$$F_c(n) = \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2}, where\ 0 < x < 4$$

Sol. From the inverse finite cosine transform, we have

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{4}.1 + \frac{2}{4}\sum_{n=1}^{\infty}F_c(n).\cos\left(\frac{n\pi x}{l}\right) = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2}\cos\left(\frac{n\pi x}{4}\right)$$